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The Taylor–Saffman problem for a non-Newtonian liquid

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The Taylor–Saffman problem concerns the fingering instability which develops when one liquid displaces another, more viscous, liquid in a porous medium, or equivalently for Newtonian liquids, in a Hele-Shaw cell. Recent experiments with Hele-Shaw cells using non-Newtonian liquids have shown striking qualitative differences in the fingering pattern, which for these systems branches repeatedly in a manner resembling the growth of a fractal. This paper is an attempt to provide the beginnings of a hydrodynamical theory of this instability by repeating the analysis of Taylor & Saffman using a more general constitutive model. In fact two models are considered; the Oldroyd 'Fluid B' model which exhibits elasticity but not shear thinning, and the Ostwald–de Waele power-law model with the opposite combination. Of the two, only the Oldroyd model shows qualitatively new effects, in the form of a kind of resonance which can produce sharply increasing (in fact unbounded) growth rates as the relaxation time of the fluid increases. This may be a partial explanation of the observations on polymer solutions; the similar behaviour reported for clay pastes and slurries is not explained by shear-thinning and may involve a finite yield stress, which is not incorporated into either of the models considered here.

1. Introduction

The Hele-Shaw cell is a well-known device consisting of two parallel glass plates separated by a narrow gap. When a viscous liquid flows in the gap the mean velocity $\bar{u}$ is related to the pressure p by the equation

$$\bar{u} = -\frac{h^2}{12\eta}\operatorname{grad} p$$

where h is the plate separation and η is the viscosity. This is the same as the equation for flow in a porous medium of permeability $\frac{1}{12}h^2$ and much of the interest in the Hele-Shaw cell arises from this fact.

When the cell contains two immiscible liquids the interface between them is unstable when the less viscous fluid displaces the more (gravitational effects being left out). This is the subject of the celebrated paper (Taylor & Saffman 1958) from which the instability derives its name. A number of questions were left open in that paper and much development has taken place; see, for example, Saffman (1986) and Homsy (1987) for recent surveys.

In the last few years attention has been drawn to novel effects which can be observed when the displaced fluid is non-Newtonian in character. Experiments in which a polymer solution is displaced by water have been described by Nittman, Daccord & Stanley (1985) and Daccord & Nittman (1986), and others using clay pastes and slurries by Van Damme *et al.* (1987, 1988) and Daccord & Lenormand (1987). Other experiments, using both ideal elastic (Boger) fluids and shear-thinning polymer solutions have been reported by Allen & Boger (1988). It was observed that the fingering pattern which develops at the interface branches repeatedly in a manner which suggests the growth of a fractal. No detailed theory of the interfacial instability has been put forward, although there has been some discussion of orders of magnitude (de Gennes 1987). The experiments were compared with computer simulations of a random walk process by Nittman *et al.* (1985) and by others; see Meakin (1987) for example.

The connection between the Hele-Shaw flow of a liquid and the flow in a porous medium, when the liquid is viscoelastic, is presumably rather remote. However, the problem is of intrinsic interest, even if the industrial relevance is absent.

This paper is an attempt to provide a hydrodynamical theory, and has the straightforward plan of repeating the stability calculations of Taylor & Saffman, using a more general constitutive model for the fluid but otherwise following their general approach as closely as the additional complications permit. In particular we use the same depth-averaged interface conditions, as explained presently.

It will be supposed that the liquid is displaced by air (that is, a fluid whose dynamics may be ignored) along an interface which is initially straight, and the stability of this interface to small disturbances will be examined. Taylor & Saffman used depth-averaged equations throughout; that is, they considered the behaviour of small disturbances to equations which had already been averaged. This is not feasible in the present case because the equations are more intricately coupled, and the procedure is as follows. The disturbance equations are derived from the full equations and solved subject to the no-slip conditions on the walls; the approximations of lubrication theory are made, on the assumption that the plate separation h is much smaller than any other lengthscale of interest. Taylor & Saffman also used depth-averaged interface conditions. Considerable effort has been put into refining this, for example by Park & Homsy (1984) and by Reinelt (1987); the conclusion is that for small capillary numbers Taylor & Saffman were not far wrong, in that a coefficient 1 should be replaced by $\frac{1}{4}\pi$ etc. In view of this, and in the absence of any corresponding accurate theory of the interface for non-Newtonian fluids, the original boundary condition of Taylor & Saffman will be used here. Thus a mean (i.e. depth-averaged) kinematic condition and stress balance are imposed. Note that the equations of motion are not themselves averaged; the relevant components of the solution are averaged and then used in the interface conditions. This closes the problem and determines the growth rate as a function of wavenumber.

Of course the representation of the flow field as basic parallel flow plus small disturbance must fail very close to the interface, probably within a distance of order h, and the same is true of Taylor & Saffman's theory. The idea is that the instability is the result of interactions on a lengthscale large compared with h, so that details near the meniscus are not important; this places a restriction on the disturbance wavelength to which the theory can be expected to apply.

Non-Newtonian liquids exhibit at least two characteristics not present in Newtonian liquids, namely elasticity and shear-thinning, and in order to disentangle these effects two constitutive models are considered, each of which possesses one property but not the other. First is the Oldroyd 'Fluid B' model which shows elasticity but constant shear viscosity (although the stress field has more non-zero components than a Newtonian liquid). Then we consider the Ostwald–de Waele power-law fluid which shows shear-thinning but is inelastic. Of the two, the Oldroyd model is more difficult and more interesting; the presence of elasticity produces the possibility of a resonance in the disturbance equations which is excited by the interface conditions. According to the theory given here, for suitable values of the model parameters the disturbance growth rate becomes unbounded. The power-law fluid behaves more or less as a Newtonian fluid; the growth rate of the disturbance is multiplied by $k^{-\frac{1}{2}}$, where k is the power-law exponent, as compared with the Newtonian case, but there is no qualitative change.

2. The Oldroyd 'Fluid B'

We choose coordinates so that the two plates of the Hele-Shaw cell correspond to $y = 0$ and $y = h$ and in the basic undisturbed flow the liquid moves in the positive x-direction. The velocity field of this undisturbed motion has the form $(u_0(y), 0, 0)$ and it is supposed that the liquid is displaced by air (or some fluid whose dynamics can be neglected) in such a way that the undisturbed interface is the straight line $x = \bar{u}_0 t$, where $\bar{u}_0$ is the mean velocity, and the liquid occupies the region $x > \bar{u}_0 t$. So as in the original investigation of Taylor & Saffman we are leaving aside any complications very close to the meniscus, including the possibility that the liquid is not completely displaced.

To determine $u_0(y)$ and the associated stress field we now consider in detail the constitutive and force balance equations. This is most conveniently done using subscript notation and the summation convention. The coordinates are denoted x_i and the velocity components u_i, and we define

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1}$$

$$w_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right). \tag{2}$$

The constitutive equation is

$$\tau_{ij} + \lambda_1\{\mathring{\tau}_{ij} - (\tau_{ik} e_{kj} + e_{ik}\tau_{kj})\} = 2\eta[e_{ij} + \lambda_2\{\mathring{e}_{ij} - 2e_{ik}e_{kj}\}], \tag{3}$$

where τ_{ij} is the extra stress tensor and $\mathring{\tau}_{ij}$ (or $\mathring{e}_{ij}$) denotes the corotational derivative,

$$\mathring{\tau}_{ij} \equiv \left\{\frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k}\right\}\tau_{ij} + w_{ik}\tau_{kj} - \tau_{ik} w_{kj}, \tag{4}$$

with a similar expression for $\mathring{e}_{ij}$. Equation (3) is also known as the upper-convected Jeffreys model; for further details see Petrie (1979) or Bird, Armstrong & Hassager (1977). We note here that for $\lambda_2 = 0$ we recover the well-known and simpler upper-convected Maxwell model; and for $\lambda_1 = \lambda_2$ we recover the Newtonian model. This latter simplification can be observed in the various appearances of equation (3) in the analysis below and provides a useful check.

The stress tensor σ_{ij} is given by

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{5}$$

and, neglecting inertial effects, the equations of motion are

$$\frac{\partial}{\partial x_j}\sigma_{ij} = 0. \tag{6}$$

For the simple unidirectional flow considered here these equations may be solved without difficulty. The pressure p is independent of y and z and the pressure gradient is a function of t only. For simplicity we shall suppose that arrangements are made to maintain a constant pressure gradient in the liquid (or equivalently, a constant mean flow) so that the pressure p_0 is given by

$$p_0 = G(x-\bar{u}_0 t) \quad (x-\bar{u}_0 t > 0) \tag{7}$$

where G is a (negative) constant. Then we find

$$u_0 = \frac{G}{\eta}(\tfrac{1}{2}y^2 - \tfrac{1}{2}hy), \tag{8}$$

which is the same as for a Newtonian liquid of viscosity η; at this point the absence of shear-thinning behaviour becomes apparent. There are two non-zero components of τ_{ij}, namely

$$(\tau_{xy})_0 \equiv T_0 = G(y-\tfrac{1}{2}h), \tag{9}$$

and

$$(\tau_{xx})_0 \equiv S_0 = \frac{2(\lambda_1-\lambda_2)}{\eta}G^2(y-\tfrac{1}{2}h)^2. \tag{10}$$

Here T_0 is again the same as for a Newtonian liquid and only the component τ_{xx} depends on the relaxation time constants λ_1 and λ_2. Finally the mean velocity $\bar{u}_0$ is given by

$$\bar{u}_0 = -Gh^2/12\eta. \tag{11}$$

We turn now to the problem of the stability of this flow. The velocities and stresses are written as the sum of the primary quantities as just obtained, which have a subscript zero, and small disturbances; then they are substituted into (1)–(6), which are linearized in the usual way. This process is lengthy but straightforward. We defer presentation of the results until a suitable choice of dimensionless variables has been found and this in turn requires an analysis of the conditions at the interface $x = \bar{u}_0 t$.

We write the equation of the perturbed interface as

$$x = \bar{u}_0 t + \xi(z,t). \tag{12}$$

The mean velocity is $\bar{u}_0 + \bar{u}$, the pressure is $p_0 + p$ and the relevant component of the extra stress is $S_0 + \tau_{xx}$. Two conditions are to be imposed at the interface (12), namely the kinematical condition and the condition of continuity of normal stress, both averaged with respect to y.

The kinematical condition, when linearized, becomes

$$\bar{u} = \frac{\partial \xi}{\partial t} \quad \text{on} \quad x = \bar{u}_0 t, \tag{13}$$

and the stress condition becomes

$$G\xi + p - \bar{\tau}_{xx} = \gamma \frac{\partial^2 \xi}{\partial z^2} \quad \text{on} \quad x = \bar{u}_0 t, \tag{14}$$

where γ is the surface tension of the liquid. (The curvature of the meniscus in the (x, y)-plane, and the mean contribution of S_0, have the effect of adding a constant to the pressure p_0, which has no effect on the stability and can be left out.) We now assume that

$$\xi \propto \cos(nz)\exp(\mu t), \tag{15}$$

$$p, \bar{u}, \bar{\tau}_{xx} \propto \cos(nz)\exp(\sigma t+\alpha x), \tag{16}$$

where n is the wavenumber of the disturbance, μ and σ are the growth rates, to be found, and α is the decay rate, which is also to be found, with the restriction $\mathrm{Re}(\alpha) < 0$ for acceptable behaviour as $x \to \infty$. From (13) we find

$$\sigma+\alpha\bar{u}_0 = \mu \tag{17}$$

and then ξ may be eliminated to give a single condition,

$$(G+\gamma n^2)\,\bar{u}+\mu(p-\bar{\tau}_{xx}) = 0 \quad \text{on} \quad x = \bar{u}_0 t. \tag{18}$$

From this equation, and from (11) which connects $\bar{u}_0$ and G, we find the appropriate lengthscale a for x and z, and the perturbation variables, given by

$$a^2 = \gamma h^2/\eta\bar{u}_0, \tag{19}$$

and the timescale is $a/\bar{u}_0$. These will be used to form a dimensionless version of (18), and also to form dimensionless versions of the constitutive and equilibrium equations which will be displayed shortly.

The choice of scales for the perturbation quantities is as follows. There will be an arbitrary constant corresponding to the amplitude of the perturbation and this is conveniently taken to be the magnitude of the perturbation to u_0, say $\hat{U}$. Then we have u, $w \sim \hat{U}$, $v \sim \hat{U}h/a$. Convenient scales for the stress components are

$$p, \tau_{xx}, \tau_{xz} \sim \eta\hat{U}a/h^2,$$
$$\tau_{xy}, \tau_{yz} \sim \eta\hat{U}/h,$$
$$\tau_{yy} \sim \eta\hat{U}/a.$$

In order to avoid cumbersome superscripts we shall now use the letters u, p, τ etc., to denote *dimensionless* variables, scaled as indicated, and accept the slight risk of confusion. So (18) becomes

$$(n^2-12)\,\bar{u}+\mu(p-\bar{\tau}_{xx}) = 0. \tag{20}$$

The constitutive equations, when linearized, involve the functions u_0, T_0 and S_0 in the coefficients, which in turn depend on y, and also derivatives with respect to y, and it is correct to scale y on h. Meanwhile we scale x and z on a as indicated above and assume that $h \ll a$. This lubrication-type approximation allows certain terms to be dropped. We introduce two Weissenberg numbers

$$W_1 = \frac{12\lambda_1\bar{u}_0}{a}, \quad W_2 = \frac{12\lambda_2\bar{u}_0}{a} \tag{21}$$

(where the factor 12 is inserted for later numerical convenience), and the operators $\mathscr{D}_1$ and $\mathscr{D}_2$, given by

$$\mathscr{D}_{1,2} = 1+\tfrac{1}{12}W_{1,2}(\sigma+6\alpha(y-y^2)). \tag{22}$$

These are the linearized versions of the operators $1+\lambda_{1,2}\,\mathrm{D}/\mathrm{D}t$, which appear in (3).

Thus we obtain

$$\mathscr{D}_1 \tau_{xx} + 4W_1(W_1 - W_2)(y - \tfrac{1}{2})v - 4W_1(W_1 - W_2)(y - \tfrac{1}{2})^2 \frac{\partial u}{\partial x} - 2W_1(\tfrac{1}{2} - y)\frac{\partial u}{\partial y}$$
$$- 2W_1(\tfrac{1}{2} - y)\tau_{xy} = -4W_2(\tfrac{1}{2} - y)\frac{\partial u}{\partial y}, \quad (23)$$

$$\mathscr{D}_1 \tau_{xy} - W_1 v - W_1(\tfrac{1}{2} - y)\frac{\partial u}{\partial x} - W_1(\tfrac{1}{2} - y)\tau_{yy} - 2W_1(W_1 - W_2)(y - \tfrac{1}{2})^2 \frac{\partial v}{\partial x}$$
$$- W_1(\tfrac{1}{2} - y)\frac{\partial v}{\partial y} = \mathscr{D}_2 \frac{\partial u}{\partial y} - W_2 v - W_2(\tfrac{1}{2} - y)\left(\frac{\partial u}{\partial x} + 3\frac{\partial v}{\partial y}\right), \quad (24)$$

$$\mathscr{D}_1 \tau_{xz} - W_1(\tfrac{1}{2} - y)\tau_{yz} - 2W_1(W_1 - W_2)(y - \tfrac{1}{2})^2 \frac{\partial w}{\partial x} - W_1(\tfrac{1}{2} - y)\frac{\partial w}{\partial y} = -2W_2(\tfrac{1}{2} - y)\frac{\partial w}{\partial y}, \quad (25)$$

$$\mathscr{D}_1 \tau_{yy} - 2W_1(\tfrac{1}{2} - y)\frac{\partial v}{\partial x} = 2\mathscr{D}_2 \frac{\partial v}{\partial y} - 2W_2(\tfrac{1}{2} - y)\frac{\partial v}{\partial x}, \quad (26)$$

$$\mathscr{D}_1 \tau_{yz} - W_1(\tfrac{1}{2} - y)\frac{\partial w}{\partial x} = \mathscr{D}_2 \frac{\partial w}{\partial y} - W_2(\tfrac{1}{2} - y)\frac{\partial w}{\partial x}. \quad (27)$$

The equilibrium equations become

$$-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{xy} + \frac{\partial}{\partial z}\tau_{xz} = 0, \quad (28)$$

$$\frac{\partial p}{\partial y} = 0, \quad (29)$$

$$-\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} = 0, \quad (30)$$

and finally there is the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (31)$$

In all the equations (23)–(31) we should of course now assume a form of disturbance similar to (16), namely

$$\left.\begin{aligned} p, \tau_{xx}, \tau_{xy}, \tau_{yy}, u, v &\propto \cos nz \exp(\sigma t + \alpha x), \\ \tau_{xz}, \tau_{yz}, w &\propto \sin nz \exp(\sigma t + \alpha x). \end{aligned}\right\} \quad (32)$$

In this way we obtain a set of ordinary differential equations, with y as the independent variable, subject to the usual no-slip conditions

$$u = v = w = 0 \quad \text{on} \quad y = 0, 1. \quad (33)$$

A numerical solution appears to be necessary and the procedure is as follows. The system is effectively of order 6, since y-derivatives can be eliminated from (say) (23), (25) and (26). We can choose the wavenumber n arbitrarily (and W_1 and W_2 of course); then the system contains two unknown parameters, σ and α. The imposition of (33) is equivalent to an equation connecting σ and α, and a second such equation arises from the interface condition (20), which uses averages of course. The two equations in the two unknowns σ and α must be solved numerically; however, the linearity of the system makes this whole numerical task fairly straightforward.

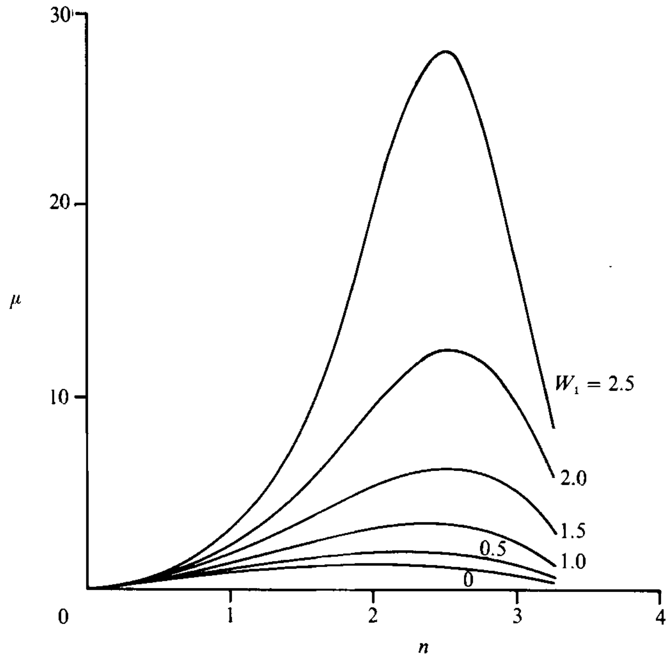

FIGURE 1. Growth rate of disturbance, μ, against wavenumber n, for various values of $W_1 = 12\lambda_1 \bar{u}_0/a$, with $W_2 = 0$. The scales for μ and n are $\bar{u}_0/a$ and $1/a$ respectively.

In the event the numerical solutions all showed a remarkable simplification; it was found in every case that

$$\alpha = -n, \quad v = 0, \quad u = w, \quad \tau_{yy} = 0, \quad \tau_{xy} = \tau_{yz}, \quad \tau_{xx} = 2\tau_{xz}. \tag{34}$$

Using (34) we can obtain a reduced set of equations,

$$\mathscr{D}_1 \tau_{xy} + (W_1 - W_2)(\tfrac{1}{2} - y)\, nu = \mathscr{D}_2 \frac{\mathrm{d}u}{\mathrm{d}y}, \tag{35}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\tau_{xy} = n\tau_{xz} - np, \tag{36}$$

$$\mathscr{D}_1 \tau_{xz} = W_1(\tfrac{1}{2} - y)\tau_{xy} - 2W_1(W_1 - W_2)(y - \tfrac{1}{2})^2 nu + (W_1 - 2W_2)(\tfrac{1}{2} - y)\frac{\mathrm{d}u}{\mathrm{d}y}, \tag{37}$$

$$\frac{\mathrm{d}p}{\mathrm{d}y} = 0, \tag{38}$$

corresponding to (23)–(30). This is still not quite enough to make an analytical solution possible but it makes the numerical solution much easier, mainly because α has been determined (equal to $-n$) and the root-finding procedure is only one-dimensional. It has not been proved that other types of solution are impossible; but none were found.

We begin by considering the special case $W_2 = 0$, corresponding to the upper-convected Maxwell model. The results are shown in figure 1. This gives graphs of the dimensionless growth rate μ (the growth rate of the interface disturbance, cf. (15))

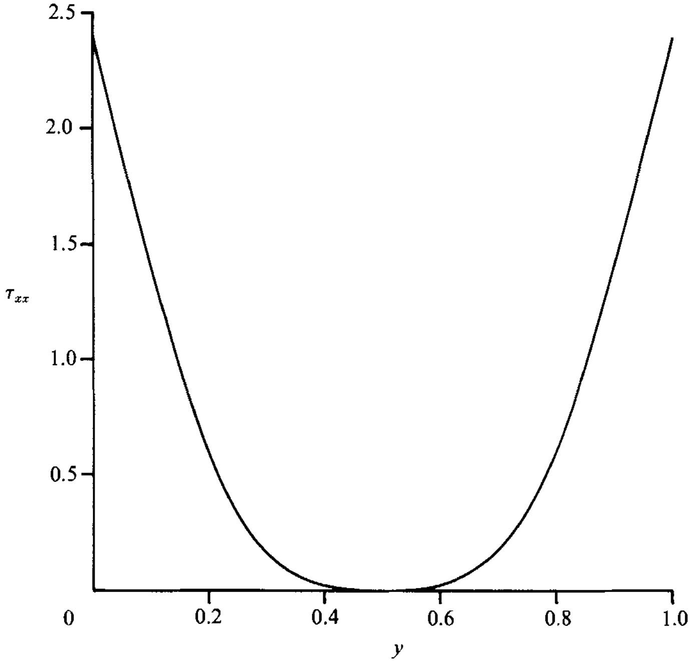

FIGURE 2. The longitudinal stress τ_{xx} against y for the case $W = 2.5, n = 2.5$. (The eigensolution has been normalized so that $p = 1$.)

against dimensionless wavenumber n, for various values of W_1. In the case $W_1 = 0$ we recover the Taylor–Saffman result

$$\mu = n - \tfrac{1}{12}n^3, \tag{39}$$

with the maximum growth rate at $n = 2$. As W_1 increases we find a slight shift in the most unstable wavenumber and a very sharp increase in maximum growth rate; this occurs at rather small values of W_1, recalling that there is a factor 12 in the definition.

This effect can be traced to a kind of resonance. Equations (35), (36) and (37) can be reduced (in this case) to

$$\left.\begin{aligned} \frac{\mathrm{d}^2u}{\mathrm{d}y^2} + 2nW_1(y-\tfrac{1}{2})\frac{\mathrm{d}u}{\mathrm{d}y} + [nW_1 + 2n^2W_1^2(y-\tfrac{1}{2})^2]\,u &= -n\mathscr{D}_1 p, \\ u(0) = u(1) &= 0. \end{aligned}\right\} \tag{40}$$

Of course this system alone is not sufficient to determine σ (or μ) since the interface condition (20) must be used. But the solution for u will become unbounded when nW_1 is close to an eigenvalue of the operator in (40); this eigenvalue (more precisely, the first eigenvalue) was found numerically to be $nW_1 = 16.05$ approximately. The combination nW_1 is independent of a and is in fact equal to $12\lambda_1 \bar{u}_0 n^*$, where n^* is the dimensional wavenumber. Thus the resonance occurs when $\lambda_1 \bar{u}_0 n^* \approx 1$ and, since $\alpha = -n$, this means that the relaxation time of the fluid is comparable with the time taken to pass through the region in which the perturbation has significant amplitude.

The values of nW_1 corresponding to the results in figure 1 are not particularly close to 16.05 (they were about 6 or 7 at most) but large values of u were obtained. For

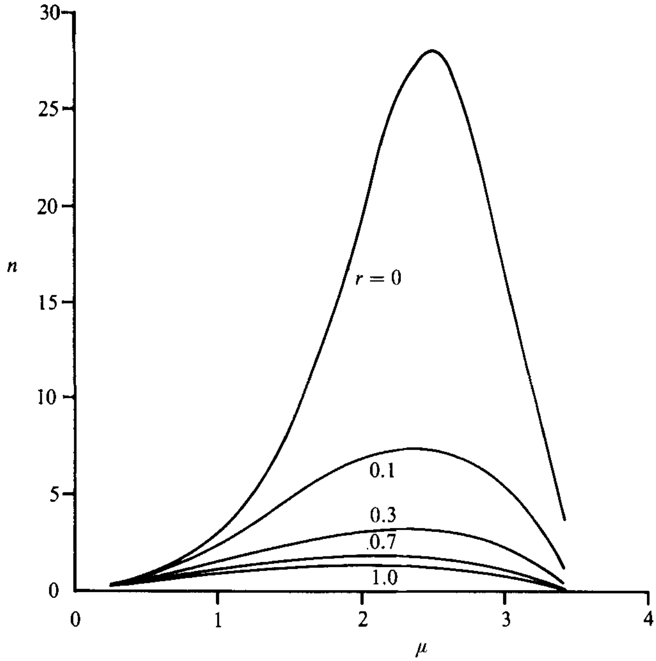

FIGURE 3. Growth rate of disturbance, μ, against wavenumber n, for various values of $r = W_2/W_1$, with $W_1 = 2.5$. Scales as in figure 1.

example at $W_1 = 2.5, n = 2.5$ it was found that u is about 8 times as large as for $W_1 = 0, n = 2.5$. This produces a large value of $\bar{u}$ and hence a large value of μ, as can be seen from (20). Another feature of the solution which has the same tendency is the appearance of the mean viscous stress term $\bar{\tau}_{xx}$ in (20). This was always positive; so this tensile stress reduces the effect of the disturbance pressure p and increases μ. In fact the numerical solutions for n close to $(12)^{\frac{1}{2}}$ sometimes showed $p-\bar{\tau}_{xx}$ changing sign; but this singularity is probably a spurious effect of the averaging process since τ_{xx} showed a boundary-layer structure in those cases, with large values near the walls and small values on the centre of the channel. An example is shown in figure 2.

Now we turn to the effect of the second time constant λ_2. When λ_2 is non-zero the growth rates are reduced and the singularity at $nW_1 \approx 16.05$ is (apparently) removed. Sample results are shown in figure 3. This shows the growth rate μ against n, with $W_1 = 2.5$, for various values of the ratio $r = W_2/W_1$. Thus the curve $r = 0$ is the same as the upper curve of figure 1, and the curve $r = 1$ (for which the Newtonian case is recovered) is the same as the lowest curve of figure 1. Further numerical experiments were carried out at fixed values of r with increasing values of W_1 with similar results. No resonance was found, although numerical difficulties were encountered at large W_1 which were associated with spurious (unphysical) solutions generated by the root-finding routine.

This may be related to the existence of a critical extension rate for the Maxwell model (Petrie 1979, p. 43). In simple uniaxial extension, the stress become infinite at a certain extension rate; the singularity is shifted, but not removed, by the incorporation of the extra terms in the Jeffreys model (or Oldroyd 'B' model). Another possibility is that the Maxwell fluid is instantaneously elastic, that is, the relaxation modulus is finite, whereas the Jeffreys (or Oldroyd) model incorporates a

Newtonian element resulting in infinite instantaneous rigidity. These ideas have been thoroughly discussed by Joseph, Narain & Riccius (1986). It is hoped to pursue these questions further in the context of a study of some simpler stability problems, where there is less uncertainty about the extent to which the results are model-dependent.

3. The Ostwald–de Waele power-law model

The procedure here is essentially the same as in the previous section except that we use the much simpler constitutive relation

$$\tau_{ij} = 2\eta e_{ij} \tag{41}$$

with

$$\eta = m(2e_{ij}\,e_{ij})^{\frac{1}{2}(k-1)}, \tag{42}$$

where m and k are constants; usually $k \leqslant 1$. The Newtonian case is recovered by setting $k = 1$.

First we solve for the basic parallel flow, and as before we can put

$$p_0 = G(x-\bar{u}_0 t) \quad (x-\bar{u}_0 t) > 0, \tag{43}$$

where G is a (negative) constant, and we find

$$u_0(y) = \left(-\frac{G}{m}\right)^{1/k} \frac{1}{K}[(\tfrac{1}{2}h)^K - |y-\tfrac{1}{2}h|^K], \tag{44}$$

where

$$K = \frac{k+1}{k}. \tag{45}$$

The mean velocity is therefore given by

$$\bar{u}_0 = \left(-\frac{G}{m}\right)^{1/k} (\tfrac{1}{2}h)^K \frac{k}{2k+1}. \tag{46}$$

Now we linearize the equations about this basic flow, using u, v, w, p etc. to denote perturbation quantities as before. It is necessary to expand (42) of course and it is convenient to denote the (variable) shear viscosity corresponding to the basic velocity field (44) by η_0, where

$$\begin{aligned}\eta_0 &= m\left|\frac{\mathrm{d}u_0}{\mathrm{d}y}\right|^{k-1} \\ &= m\left|\frac{G}{m}(y-\tfrac{1}{2}h)\right|^{(k-1)/k}.\end{aligned} \tag{47}$$

After a lengthy reduction and after use of the lubrication approximation, the equations of motion can be written

$$-\frac{\partial p}{\partial x} + k\left(\eta_0\frac{\partial^2 u}{\partial y^2} + \frac{\mathrm{d}\eta_0}{\mathrm{d}y}\frac{\partial u}{\mathrm{d}y}\right) = 0, \tag{48}$$

$$-\frac{\partial p}{\partial y} = 0, \tag{49}$$

$$-\frac{\partial p}{\partial z} + \eta_0\frac{\partial^2 w}{\partial y^2} + \frac{\mathrm{d}\eta_0}{\mathrm{d}y}\frac{\partial w}{\partial y} = 0. \tag{50}$$

The boundary conditions are

$$u = w = 0 \quad \text{on} \quad y = 0, h \tag{51}$$

and the interface condition (14) holds, although τ_{xx} is zero in the present case. We can look for solutions having the forms

$$\left.\begin{aligned} p &= P \\ u &= u(y) \end{aligned}\right\} \cos nz \exp(\sigma t + \alpha x), \qquad w = w(y) \sin nz \exp(\sigma t + \alpha x), \tag{52}$$

where P is a constant. Then (48) and (50) can be solved to give

$$u = \frac{\alpha P}{k} f(y), \quad w = -nPf(y),$$

$$f(y) = \frac{1}{m}\left|\frac{m}{G}\right|^{(k-1)/k} \frac{1}{K}\left[(\tfrac{1}{2}h)^K - |y - \tfrac{1}{2}h|^K\right]. \tag{53}$$

A connection between α and n may be established by means of the averaged continuity equation

$$\alpha\bar{u} + n\bar{w} = 0, \tag{54}$$

the overbar denoting a mean value over $0 \leqslant y \leqslant h$. Thus $\alpha^2 = kn^2$ or, since we require $\alpha < 0$ for solutions decaying at $x \rightarrow \infty$,

$$\alpha = -nk^{\frac{1}{2}}. \tag{55}$$

It remains to evaluate $\bar{u}$ and substitute into (18) (with $\bar{\tau}_{xx} = 0$) to determine the growth rate μ. We can make this dimensionless in the same way as in the previous section, using the formula (46) for $\bar{u}_0$ instead of (11) to define the length- and timescales, and there results

$$\mu = k^{-\frac{1}{2}} n(1 - \tfrac{1}{12}n^2), \tag{56}$$

where the variables are now dimensionless. This may be compared with (39) which is of course recovered on setting $k = 1$.

4. Concluding remarks

No direct comparison between the experiments and the present theory is possible because very little is known about the rheological behaviour of the fluids discussed, and the constitutive models used here may have little resemblance to the real thing. The clay pastes are considered by Van Damme *et al.* (1987) to be shear thinning with a yield stress, and estimates of the parameters are given. They are also described as elastic, although this seems unlikely, and no relaxation time is given. The power-law model considered here gives no hint of the behaviour observed by Van Damme *et al.* (1987) or Allen & Boger (1988); this may be due to the assumption of zero yield stress in (42). A non-zero yield stress may well give markedly different results because ahead of the more slowly advancing parts of the interface the fluid could actually be brought to rest. But it is hard to see how this could be dealt with by a linearized theory, since in the undisturbed flow the tangential stress near the walls would have to exceed the yield stress by a finite amount, and this would still hold after an infinitesimal disturbance. More effort should no doubt be put into this line of inquiry.

W_1	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4	4.5	5
μ	1.0	1.23	1.54	1.97	2.58	3.46	4.78	6.91	10.91	21.10	105.06

TABLE 1. Growth rate μ against W_1 for the case of zero surface tension. The scale for μ is $\bar{u}_0 n$ where n is the dimensional wavenumber, and $W_1 = 12\lambda_1 \bar{u}_0 n$

The systems involving water and aqueous polymer solutions (Nittman *et al.* 1985) had (deliberately) zero or very low interfacial tension, for which the scalings of the previous sections are not appropriate, because $a = 0$. The only available lengthscale, when $\gamma = 0$, is the wavelength of the disturbance. It is not difficult to rescale the equations, using n^{-1} as the unit of length, and the effect is to produce a system of equations identical to (35)–(38) except that n is replaced by 1 and the Weissenberg numbers $W_{1,2}$ are defined by

$$W_{1,2} = 12\lambda_{1,2}\bar{u}_0 n.$$

The interface condition becomes

$$-12\bar{u}+\mu(p-\bar{\tau}_{xx}) = 0.$$

The value of the growth rate μ against W_1 for the case $W_2 = 0$ is indicated in table 1. For $W_1 = 0$ we recover the Taylor–Saffman result $\mu = 1$, but as W_1 increases there is a very sharp increase in μ. This is the counterpart of figure 1.

It may be that this partially explains the rapid fingering observed by Nittman *et al.* (1985). The validity of the lubrication approximation and of the averaging leading to (20) require that $nh/2\pi \ll 1$ and so no explanation of very fine structure is possible; however, in the experiments this condition appears to be satisfied.

Recent experiments using two Newtonian liquids, reported by Chen (1989), show marked differences in the fingering pattern according to whether the liquids are miscible or not. The branching occurs on a much finer scale for miscible systems, which have zero interfacial tension, as might be expected. On the other hand Allen & Boger (1988) conclude that the distinctive fingering pattern is due to shear thinning, since the Boger fluids did not differ greatly in their response from Newtonian fluids of similar shear viscosity.

In the light of these observations and the present calculations we might conclude that a major influence on the fingering pattern is the presence or absence of significant interfacial tension. Shear thinning appears to have no great effect although a more refined model than the present one may be required, as noted above. An elastic liquid need not differ greatly from a Newtonian liquid, but may show a marked increase in the growth rate of disturbances (but not much effect on the wavelength) if, for the Oldroyd 'B' fluid, W_1 is large and W_2 is small. This means that fluids which have similar values of the steady shear viscosity but differ in their instantaneous rigidity should behave differently; this interesting effect should be observable experimentally.

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